

# Notes on Emergent Gravity

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**ABSTRACT:** Emergent gravity is aimed at constructing a Riemannian geometry from  $U(1)$  gauge fields on a noncommutative spacetime. But this construction can be inverted to find corresponding  $U(1)$  gauge fields on a (generalized) Poisson manifold given a Riemannian metric  $(M, g)$ . We examine this bottom-up approach with the LeBrun metric which is the most general scalar-flat Kähler metric with a  $U(1)$  isometry and contains the Gibbons-Hawking metric, the real heaven as well as the multi-blown up Burns metric which is a scalar-flat Kähler metric on  $\mathbb{C}^2$  with  $n$  points blown up. The bottom-up approach clarifies some important issues in emergent gravity.

**KEYWORDS:** Models of Quantum Gravity, Gauge-Gravity Correspondence, Non-Commutative Geometry.

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## 1. Introduction

Recently the correspondence between noncommutative (NC) U(1) gauge theory and gravity has evolved at large in the context of emergent gravity. See [1, 2, 3, 4, 5, 6, 7] for recent reviews. The idea of emergent gravity is the following: Suppose that U(1) gauge theory is defined on a symplectic manifold  $(M, B)$  where  $B$  is a nondegenerate, closed two-form on a smooth manifold  $M$ . Indeed one can consider the symplectic two-form  $B$  as a field strength of vacuum gauge fields which take the form  $A_\mu^{(0)} = -\frac{1}{2}B_{\mu\nu}y^\nu$  on a local Darboux chart. Let us introduce dynamical gauge fields fluctuating around the backround  $B = dA^{(0)}$ . The resulting field strength is given by  $\mathcal{F} = B + F$  where  $F = dA$  is the curvature two-form of the dynamical gauge field  $A$ . Note that  $d\mathcal{F} = 0$  due to the Bianchi identity  $dF = 0$  and  $\mathcal{F}$  is invertible unless  $\det(1 + B^{-1}F) = 0$ . Therefore  $\mathcal{F} = B + F$  is again a symplectic structure on  $M$  and so the dynamical gauge fields defined on a symplectic vacuum  $B$  manifest themselves as a deformation of the symplectic structure [8, 9].

One may introduce local coordinates  $X^a$ ,  $a = 1, \dots, 4$ , on a local chart  $U \subset M$  where the symplectic structure  $\mathcal{F}$  is represented by

$$\mathcal{F} = \frac{1}{2} \left( B_{ab} + F_{ab}(X) \right) dX^a \wedge dX^b. \quad (1.1)$$

But one can introduce another coordinates, say  $y^\mu$ , on the same local patch  $U \subset M$  which are diffeomorphic to  $X^a$ , i.e.  $X^a = X^a(y)$ . Now one can ask an interesting question whether it is possible to find a coordinate transformation  $f : X \mapsto y = y(X)$  in order to eliminate the electromagnetic force  $F = dA$  in the symplectic structure  $\mathcal{F} = B + F$ . In

other words, one may try to find a local coordinate transformation  $f : X \mapsto y = y(X)$  such that the symplectic structure  $\mathcal{F}$  in (1.1) on  $U \subset M$  becomes

$$\mathcal{F}|_U = \frac{1}{2} B_{\mu\nu} dy^\mu \wedge dy^\nu. \quad (1.2)$$

Remarkably, the Darboux theorem or the Moser lemma in symplectic geometry [10, 11] says that it is always possible to find such a local coordinate transformation as long as the space  $M$  admits a symplectic structure. If so, it is immediate to see from (1.1) that the so-called Darboux coordinates  $y^\mu$  will obey the following relation [12, 13]

$$\left( B_{ab} + F_{ab}(X) \right) \frac{\partial X^a}{\partial y^\mu} \frac{\partial X^b}{\partial y^\nu} = B_{\mu\nu}. \quad (1.3)$$

By taking the inverse of (1.3), one can rewrite it in the form

$$\Theta^{ab}(X) \equiv \left( \frac{1}{B + F} \right)^{ab} (X) = \theta^{\mu\nu} \frac{\partial X^a}{\partial y^\mu} \frac{\partial X^b}{\partial y^\nu} \equiv \{X^a, X^b\}_\theta(y) \quad (1.4)$$

where  $\theta \equiv \left( \frac{1}{B} \right) = \frac{1}{2} \theta^{\mu\nu} \frac{\partial}{\partial y^\mu} \wedge \frac{\partial}{\partial y^\nu} \in \Gamma(\wedge^2 TM)$  is a bivector field that defines a Poisson structure on  $M$ . The Poisson structure defines an  $\mathbb{R}$ -bilinear operation  $\{-, -\}_\theta$ , the so-called Poisson bracket [10, 11], given by

$$(f, g) \mapsto \{f, g\}_\theta = \theta(df, dg) = \theta^{\mu\nu} \frac{\partial f}{\partial y^\mu} \frac{\partial g}{\partial y^\nu} \quad (1.5)$$

for smooth functions  $f, g$ . Let us represent the coordinate transformation in the following form

$$X^a(y) = y^a + \theta^{ab} \hat{A}_b(y). \quad (1.6)$$

Then (1.4) reads as [14, 15, 16]

$$\Theta^{ab}(X) = \left( \theta - \theta \hat{F} \theta \right)^{ab} (y) \quad \Leftrightarrow \quad \hat{F}_{\mu\nu}(y) = \left( \frac{1}{1 + F\theta} F \right)_{\mu\nu} (X) \quad (1.7)$$

where

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + \{\hat{A}_\mu, \hat{A}_\nu\}_\theta. \quad (1.8)$$

Once we know fluctuations described by  $F = dA$ , we can, in principle, solve (1.7), known as the Seiberg-Witten map [17], to find the coordinate transformation (1.6) that locally eliminates the electromagnetic force  $F = dA$ .

In the end we have arrived at an important result [8, 9] that the electromagnetic force can always be eliminated by a local coordinate transformation as long as U(1) gauge theory is defined on a symplectic manifold  $M$  with symplectic structure  $B$ . In other words, there exists an analogue of the equivalence principle even for the electromagnetic force whenever U(1) gauge fields have a vacuum condensate  $\langle \mathcal{A}_\mu(y) \rangle_{\text{vac}} \equiv A_\mu^{(0)}(y) = -\frac{1}{2} B_{\mu\nu} y^\nu$ .<sup>1</sup>

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<sup>1</sup>As will be shown later, the equivalence principle for the electromagnetic force guarantees that gravity can emerge from NC U(1) gauge theory [9]. It turns out that the emergent gravity from NC U(1) gauge fields can be formulated in a background independent way where no spacetime structure is assumed but defined by the theory itself [4, 5]. Therefore one should not interpret the vacuum gauge field  $A_\mu^{(0)}(y) = -\frac{1}{2} B_{\mu\nu} y^\nu$  as an extra background condensed on a pre-existing spacetime. The flat spacetime (with Lorentz symmetry as an isometry) will emerge as a result of the vacuum condensate and hence it does not break the Lorentz symmetry [9].

Consequently, U(1) gauge theory on a symplectic manifold  $(M, B)$  boils down to solving the Seiberg-Witten map (1.7). If one has successfully solved (1.7) to determine  $\hat{A}_\mu(y)$  (which will be identified with NC U(1) gauge fields after quantization), all (at least local) informations of electromagnetic fields on the symplectic vacuum  $B$  are encoded into the coordinate transformation (1.6).

Since the coordinates  $X^a(y)$  can be regarded as smooth functions on  $M$  and they are defined on a Poisson manifold  $(M, \theta)$  as was already implied by (1.4), one can define an adjoint operation in the Poisson algebra:

$$V_a(f) = \{C_a, f\}_\theta \quad (1.9)$$

where  $f(y)$  is a smooth function and

$$C_a(y) \equiv B_{ab}X^b(y) = B_{ab}y^b + \hat{A}_a(y) \quad (1.10)$$

will be dubbed as “symplectic gauge fields.”<sup>2</sup> The adjoint operation (1.9) satisfies the Leibniz rule, i.e.,

$$\{C_a, f \cdot g\}_\theta = \{C_a, f\}_\theta \cdot g + f \cdot \{C_a, g\}_\theta \quad (1.11)$$

for any functions  $f, g$  and thus  $V_a$ ’s can be regarded as derivations. In particular,  $V_a$  can be identified with vector fields on tangent bundle  $TM \rightarrow M$ , that is,  $V_a \in \Gamma(TM)$ . Since the U(1) gauge fields  $A_\mu(X)$  are encoded into the coordinate transformations  $\hat{A}_\mu(y)$  via the Darboux theorem and then mapped to vector fields in (1.9), the U(1) gauge theory on symplectic manifold  $(M, B)$  can now be transformed into some geometry described by the vector fields  $V_a$  [3].

In terms of local coordinates  $y^\mu$  on a Darboux chart  $U \subset M$ , the Hamiltonian vector fields  $V_a \in \Gamma(TM)$  are given by

$$V_a = V_a^\mu(y) \frac{\partial}{\partial y^\mu} \quad \text{with} \quad V_a^\mu(y) = -\theta^{\mu\nu} \frac{\partial C_a(y)}{\partial y^\nu}. \quad (1.12)$$

The emergent gravity is defined by identifying a map from the vector fields in (1.12) to a gravitational metric given by

$$ds^2 = g_{\mu\nu}(x) dx^\mu \otimes dx^\nu = e^a \otimes e^a. \quad (1.13)$$

This formulation of emergent gravity to define a gravitational metric from symplectic gauge fields in (1.10) will be called the top-down approach in comparison with the bottom-up approach to identify symplectic gauge fields from a given gravitational metric. In this paper we want to address the bottom-up approach of emergent gravity. In this respect, we want to emphasize that the coordinates  $y^\mu$  are Darboux coordinates satisfying the relation (1.3). But the metric (1.13) has to respect the general covariance and is represented in a general

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<sup>2</sup>We have observed in (1.3) that  $X^a$ ’s in (1.6) arise as a local trivialization of line bundle  $L \rightarrow U$  over a Darboux chart  $U$ . Thus one can regard the symplectic gauge field  $C_a$  as a local section of the line bundle  $L$  (or more precisely, a sheaf of local functions).

coordinate system, denoted by  $x^\mu$ , which is not necessarily in the Darboux frame.<sup>3</sup> One might recall that there are many different coordinate systems to represent the same metric. For example, the usual spherical coordinate representation of Eguchi-Hanson metric [18] is equivalent to the two-center Gibbons-Hawking metric [19] by a coordinate transformation [20] though their bare appearance looks very different. Therefore, in order to identify a gravitational metric from the vector fields  $V_a$ , it is convenient to first perform a general coordinate transformation from  $y^\mu$  to  $x^\mu$ , i.e.  $y^\mu \mapsto x^\mu = x^\mu(y) \in \text{Diff}(M)$  and represent the Poisson algebra  $\mathfrak{P}(M) = (C^\infty(M), \{-, -\}_\theta)$  in such a coordinate system [9]. In order to clarify this point, let us rewrite the Poisson bracket in (1.5) in the general coordinate system  $\{x^\mu\}$ :

$$\begin{aligned} \{f, g\}_\theta &= \theta^{\mu\nu} \frac{\partial f}{\partial y^\mu} \frac{\partial g}{\partial y^\nu} = \theta^{\mu\nu} \frac{\partial x^\rho}{\partial y^\mu} \frac{\partial x^\sigma}{\partial y^\nu} \frac{\partial f}{\partial x^\rho} \frac{\partial g}{\partial x^\sigma} \\ &= \{x^\mu, x^\nu\}_\theta \frac{\partial f}{\partial x^\mu} \frac{\partial g}{\partial x^\nu} = \Theta^{\mu\nu} \frac{\partial f}{\partial x^\mu} \frac{\partial g}{\partial x^\nu} = \{f, g\}_\Theta \end{aligned} \quad (1.14)$$

where

$$\Theta^{\mu\nu}(x) \equiv \{x^\mu, x^\nu\}_\theta \quad (1.15)$$

is the Poisson structure in the coordinate system  $\{x^\mu\}$ . Indeed the definition (1.15) reduces to (1.4) if one identifies  $x^\mu = X^\mu$  and  $x^\nu = X^\nu$  and so the identity (1.14) is a Poisson algebra version of the Darboux theorem (1.3).

Let us define a vector field  $X_f$  for a smooth function  $f \in C^\infty(M)$  in the general coordinate system  $\{x^\mu\}$  by

$$X_f(g) \equiv \{f, g\}_\Theta \quad \Leftrightarrow \quad X_f^\mu(x) = -\Theta^{\mu\nu}(x) \frac{\partial f(x)}{\partial x^\nu}. \quad (1.16)$$

Also we define a two-form

$$\Omega = \frac{1}{2} \Omega_{\mu\nu}(x) dx^\mu \wedge dx^\nu \quad (1.17)$$

uniquely determined by the Poisson tensor  $\Theta^{\mu\nu}(x) = (\Omega^{-1})^{\mu\nu}(x)$ . The transformed Poisson bracket is then represented by

$$\{f, g\}_\Theta = \Omega(X_f, X_g) = X_f(g) = -X_g(f) \quad (1.18)$$

for  $f, g \in C^\infty(M)$ . The identity (1.14) immediately shows that the Poisson algebra  $\mathfrak{P}(M) = (C^\infty(M), \{-, -\}_\Theta)$  has to obey the Jacobi identity. It requires  $\Omega$  to be a closed two-form, i.e.  $d\Omega = 0$  because of the identity

$$\begin{aligned} & \{\{f, g\}_\Theta, h\}_\Theta + \{\{g, h\}_\Theta, f\}_\Theta + \{\{h, f\}_\Theta, g\}_\Theta \\ &= -\left(X_h(\Omega(X_f, X_g)) + X_f(\Omega(X_g, X_h)) + X_g(\Omega(X_h, X_f))\right) \\ &= -d\Omega(X_f, X_g, X_h) = 0. \end{aligned} \quad (1.19)$$

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<sup>3</sup>For this reason, we will explicitly distinguish Darboux coordinates  $y^\mu$  in gauge theory and general coordinates  $x^\mu$  appearing in a gravitational metric.

In other words,  $(M, \Omega)$  is also a symplectic manifold.<sup>4</sup> Hence  $\mathfrak{P}(M) = (C^\infty(M), \{-, -\}_\Theta)$  is also a Lie algebra (called the Poisson-Lie algebra of  $(M, \Omega)$ ) and the mapping  $\mathfrak{H} : \mathfrak{P}(M) \rightarrow \mathfrak{X}(M)$  (where  $\mathfrak{X}(M)$  is the Lie algebra of vector fields of  $M$ ) defined by  $f \mapsto X_f$  is a Lie algebra homomorphism [10, 11], i.e.,

$$X_{\{f,g\}_\Theta} = [X_f, X_g]. \quad (1.20)$$

According to (1.16), the vector fields in a general coordinate system for the symplectic gauge fields in (1.10) are defined by

$$V_a = V_a^\mu(x) \frac{\partial}{\partial x^\mu} \quad \text{with} \quad V_a^\mu(x) = -\Theta^{\mu\nu}(x) \frac{\partial C_a(x)}{\partial x^\nu} \quad (1.21)$$

and the symplectic gauge fields are assumed to take the form

$$C_a(x) = \Omega_{ab}(x)x^b + \hat{\mathcal{A}}_a(x). \quad (1.22)$$

Hence the Hamiltonian vector fields in (1.12) can be transformed into the vector fields in (1.21) by a general coordinate transformation [9]:

$$V_a^\mu(x) = \frac{\partial x^\mu}{\partial y^\nu} V_a^\nu(y). \quad (1.23)$$

Note that the components of the vector field  $V_a \in \Gamma(TM)$  can be written in an inspiring form

$$\begin{aligned} V_a^\mu(x) &= V_a(x^\mu) = \{C_a, x^\mu\}_\theta(y) \\ &= \frac{\partial x^\mu(y)}{\partial y^a} + \{\hat{A}_a, x^\mu\}_\theta(y) \\ &\equiv D_a x^\mu, \end{aligned} \quad (1.24)$$

where both  $C_a(y)$  and  $x^\mu(y)$  are regarded as functions of the Darboux coordinates  $y^\mu$ . It has to be noted that the vector fields  $V_a$  in (1.21) are not necessarily divergence-free, i.e.  $\partial_\mu V_a^\mu = -\frac{\partial \Theta^{\mu\nu}(x)}{\partial x^\mu} \frac{\partial C_a(x)}{\partial x^\nu} \neq 0$ , although the vector fields in a Darboux frame defined by (1.12) are divergence-free. It should be the case as the divergence-free condition is not covariant under general coordinate transformations. Therefore the vector fields  $V_a$  in a general coordinate system generate longitudinal as well as transverse components altogether.<sup>5</sup> So they can be related to a basis of orthonormal tangent vectors  $E_a = E_a^\mu \partial_\mu \in$

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<sup>4</sup>This symplectic manifold can be understood as follows. Consider an arbitrary split of the electromagnetic field,  $F = F_1 + F_2$ , and suppose  $\Omega = B + F_1$  to be a primitive symplectic structure on  $M$ . One can consider a coordinate transformation  $\phi \in \text{Diff}(M)$  such that  $\phi^*(\Omega + F_2) = \Omega$ . Then  $F_1 = 0$  where  $\Omega = B$  recovers (1.3) whereas  $F_2 = 0$  where  $\phi = \text{identity}$  corresponds to commutative gauge theory. It is also straightforward to check that  $\{X^\mu, X^\nu\}_\Theta = \left(\frac{1}{\Omega + F_2}\right)^{\mu\nu} = \{X^\mu, X^\nu\}_\theta$ .

<sup>5</sup>One of us (HSY) wants to confess that he did not clearly recognize this fact before. Regrettably, in some previous works [3, 8, 9], Hamiltonian vector fields had been partially expressed in the Darboux frame like (1.12). But, note that the form invariance (1.14) of Poisson brackets under a general coordinate transformation corresponds to the diffeomorphism symmetry in general relativity. In other words, one can choose a Darboux frame with impunity to formulate the emergent gravity in the top-down approach. ("It is impossible to study this remarkable theory with experiencing at times the strange feeling that the equations and formulas somehow have a proper life, that they are smarter than we, smarter than the author himself, and that we somehow obtain from them more than was originally put into them". – Heinrich Hertz)

$\Gamma(TM)$  and cotangent vectors (vierbeins or tetrads)  $e^a = e^a_\mu dx^\mu \in \Gamma(T^*M)$  by [21, 22, 23, 9]

$$V_a = \lambda E_a \in \Gamma(TM), \quad e^a = \lambda v^a \in \Gamma(T^*M) \quad (1.25)$$

with  $\lambda \in C^\infty(M)$  to be determined. In the next section we will explain how to determine  $\lambda$  from symplectic gauge fields. The gravitational metric emergent from symplectic gauge fields in (1.10) is thus given by

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu = e^a \otimes e^a \\ &= \lambda^2 v^a \otimes v^a = \lambda^2 v^a_\mu v^a_\nu dx^\mu dx^\nu. \end{aligned} \quad (1.26)$$

As will be shown in the next section, the equations of motion for U(1) gauge fields  $\hat{A}_\mu(y)$  over the symplectic vacuum  $\langle C_a \rangle_{\text{vac}} = B_{ab} y^b$  will be transformed to the gravitational field equations for the metric (1.26) [3, 4]. This completes the idea to construct Einstein gravity from U(1) gauge fields on a symplectic manifold  $(M, B)$ , which we call top-down formulation to compare with the bottom-up approach being the main theme of this paper.

Now we want to invert the procedure of emergent gravity to find corresponding U(1) gauge fields on a Poisson manifold given a Riemannian metric  $(M, g)$ . Suppose that a Riemannian metric  $(M, g)$  is given. One can determine  $\lambda$  by solving (2.6) and then the vector fields  $V_a$  are determined by (1.25). After that, the vector fields  $V_a$  are mapped to symplectic gauge fields  $C_a(y)$  as the system of D-module which is characterized by (1.24) and provides enough data to deduce the equations of motion for symplectic gauge fields. We will examine this bottom-up approach with the LeBrun metric [24] that is the most general scalar-flat Kähler metric with a U(1) isometry and contains the Gibbons-Hawking metric [19], the real heaven [25, 26] as well as the multi-blown up Burns metric which is a scalar-flat Kähler metric on  $\mathbb{C}^2$  with  $n$  points blown up. (See [27, 28] for the Burns metric on the blow-up of  $\mathbb{C}^2$  at the origin.) The bottom-up approach clarifies some important issues in emergent gravity, one of which was already stated in footnote 5.

The paper is organized as follows. In section 2, we review how Einstein gravity arises from the emergent metric (1.26). Especially, we explain in detail how to determine  $\lambda$  in the top-down and bottom-up approaches. In section 3, we digest the most general scalar-flat Kähler metric with a U(1) isometry constructed by LeBrun [24]. We will summarize essential ingredients of the LeBrun metric for the bottom-up approach. In section 4, we specify symplectic gauge fields obtained from the LeBrun metric and derive the equations of motion for corresponding U(1) gauge fields. Finally, in section 5, we conclude the bottom-up approach of emergent gravity with several remarks and some open issues.

## 2. Emergent gravity

Let us recapitulate the underlying idea of emergent gravity. When U(1) gauge fields have a vacuum condensate  $\langle \mathcal{A}_\mu \rangle_{\text{vac}} \equiv A_\mu^{(0)}$  which admits a symplectic structure  $B = dA^{(0)}$  on the vacuum, the fluctuations of dynamical gauge fields  $A_\mu$  will be superposed on the vacuum gauge field  $A_\mu^{(0)}$  to yield  $\mathcal{A}_\mu = A_\mu^{(0)} + A_\mu$ . Therefore the electromagnetic force  $F = d\mathcal{A}$  manifests itself only as the deformation of underlying symplectic structure  $B$  because the

total field strength is now given by  $\mathcal{F} = d\mathcal{A} = dA^{(0)} + dA = B + F$  and hence  $d\mathcal{F} = 0$ . Since  $F = dA$  describes a fluctuation around the vacuum  $B$  and  $F \rightarrow 0$  at an asymptotic infinity, one can safely assume that  $\mathcal{F}$  is nondegenerate everywhere. Hence one can conclude that  $(M, \mathcal{F})$  defines a (dynamical) symplectic manifold. Then the Darboux theorem in symplectic geometry implies that there always exists a coordinate transformation on a local Darboux chart  $U \subset M$  to locally eliminate the electromagnetic force  $F = dA$  on  $U$ . This novel form of the equivalence principle for the electromagnetic force implies [9, 3, 4] that the electromagnetism describing a dynamical symplectic manifold  $(M, \mathcal{F})$  corresponds to a geometry of spacetime manifold  $M$  whose metric is given by (1.26).

In the top-down approach described above, one can calculate the vector fields  $V_a$  defined by (1.12) or (1.21) after a general coordinate transformation only if symplectic gauge fields  $C_a(y)$  are known. However one has to know  $\lambda \in C^\infty(M)$  in order to completely determine the metric (1.26) from symplectic gauge fields in (1.10). We will explain in detail how to determine  $\lambda$  when the vector fields  $V_a$  are known.

First let us define the covariant divergence of inverse vierbein  $E_a^\mu$  by

$$\begin{aligned}\nabla_\mu E_a^\mu &= \partial_\mu E_a^\mu + \Gamma_{\mu\nu}^\mu E_a^\nu \\ &= \partial_\mu E_a^\mu + E_a^\mu \partial_\mu \log \det e_\nu^a \\ &= -\omega_{bab} \equiv -\phi_a\end{aligned}\tag{2.1}$$

where  $\Gamma_{\mu\nu}^\rho$  and  $\omega_\mu{}^a{}_b$  are the Levi-Civita and spin connections in general relativity, respectively, and we used the well-known relation  $\Gamma_{\mu\nu}^\mu = \partial_\nu \log \sqrt{\det g_{\mu\nu}} = \omega_\mu{}^a{}_b E_a^\mu e_\nu^b + E_a^\mu \partial_\mu e_\nu^a$ . Let us introduce the structure equation for vector fields  $E_a = E_a^\mu \partial_\mu \in \Gamma(TM)$  defined by

$$[E_a, E_b] = -f_{ab}{}^c E_c\tag{2.2}$$

where the structure coefficients are given by

$$f_{ab}{}^c = E_a^\mu E_b^\nu (\partial_\mu e_\nu^c - \partial_\nu e_\mu^c).\tag{2.3}$$

After imposing the torsion-free condition,  $T^a = de^a + \omega^a{}_b \wedge e^b = 0$ , the spin connection  $\omega_\mu{}^a{}_b$  can be completely determined in terms of the structure coefficients in (2.3) as

$$\omega_{abc} = E_a^\mu \omega_{\mu bc} = \frac{1}{2}(f_{abc} - f_{bca} + f_{cab}).\tag{2.4}$$

From either (2.3) or (2.4), one can easily derive the relation  $\omega_{bab} = f_{bab} = \phi_a$ , i.e.,

$$\nabla \cdot E_a = -f_{bab} = -\phi_a.\tag{2.5}$$

As was rigorously shown in [29] (see also [9]), by performing a local  $\text{SO}(4)$  rotation of basis vectors  $E_a$ , one can always achieve the gauge condition  $\phi_a = -E_a \log \lambda$  and so

$$\nabla \cdot E_a = -\phi_a = E_a \log \lambda.\tag{2.6}$$

This means [22] that one can choose  $\lambda$  by a local frame rotation such that the vector field  $E_a$  preserves the volume form  $\tilde{\nu} = \lambda^{-1} \nu_g$  where  $\nu_g = e^1 \wedge \cdots \wedge e^4 = \sqrt{\det g_{\mu\nu}} d^4x$  is the



Riemannian volume form. This can be checked as follows:

$$\begin{aligned}
\mathcal{L}_{E_a} \tilde{\nu} &= d\iota_{E_a} \left( \lambda^{-1} \det e_\mu^a dx^\mu \right) \\
&= d \left( \lambda^{-1} \det e_\nu^a \sum_{\mu=1}^4 (-1)^{\mu-1} E_a^\mu dx^1 \wedge \cdots \wedge \widehat{dx^\mu} \wedge \cdots \wedge dx^4 \right) \\
&= \left( \partial_\mu E_a^\mu + E_a^\mu \partial_\mu \log \det e_\nu^a - E_a \log \lambda \right) \tilde{\nu} \\
&= \left( \nabla \cdot E_a - E_a \log \lambda \right) \tilde{\nu} = 0
\end{aligned} \tag{2.7}$$

where  $\widehat{dx^\mu}$  denotes the omission of  $dx^\mu$ . In the above calculation, we used the Cartan's homotopy formula [10, 11]

$$\mathcal{L}_X = d\iota_X + \iota_X d \tag{2.8}$$

for Lie derivative  $\mathcal{L}_X$  along a vector field  $X \in \Gamma(TM)$  which is an important formula in differential geometry. Given the relation (1.25), the equation (2.7) suggests that the vector field  $V_a$  preserves the volume form  $\nu = \lambda^{-2} \nu_g = \lambda^2 v^1 \wedge \cdots \wedge v^4$ , which will be called the symplectic volume form, due to the relation [22]

$$0 = \mathcal{L}_{E_a} \tilde{\nu} = \mathcal{L}_{\lambda^{-1} V_a} \tilde{\nu} = \mathcal{L}_{V_a} (\lambda^{-1} \tilde{\nu}) = \mathcal{L}_{V_a} \nu. \tag{2.9}$$

The above equation means (by the same calculation as (2.7)) that

$$\begin{aligned}
\mathcal{L}_{V_a} \nu &= \left( \partial_\mu V_a^\mu + V_a^\mu \partial_\mu \log \det v_\nu^a + 2V_a \log \lambda \right) \nu \\
&= \left( \nabla \cdot V_a + 2V_a \log \lambda \right) \nu \\
&= \left( -g_{bab} + 2V_a \log \lambda \right) \nu = 0.
\end{aligned} \tag{2.10}$$

In the last step of (2.10), we have introduced the structure equation for the vector fields  $V_a$  defined by

$$[V_a, V_b] = -g_{ab}^c V_c. \tag{2.11}$$

In the top-down approach, on one hand, we know  $V_a \in \Gamma(TM)$  by (1.12) derived from symplectic gauge fields given by (1.10). Thus one can solve (2.10) to determine  $\lambda$  and hence determine the Riemannian metric (1.26) using the relation (1.25). In this way, one can completely determine the gravitational metric (1.26) emergent from U(1) gauge fields. In the bottom-up approach, on the other hand, we know a metric  $(M, g)$  instead, i.e.  $E_a \in \Gamma(TM)$ . Then one can solve (2.7) to determine  $\lambda$  and so the vector fields  $V_a \in \Gamma(TM)$  are determined by (1.25). When  $\lambda$  is known, one can also construct the symplectic volume form  $\nu = \lambda^2 v^1 \wedge \cdots \wedge v^4$  which leads to the relation [9]

$$\lambda^2 = \nu(V_1, \cdots, V_4). \tag{2.12}$$

After determining  $V_a$ 's, one can try to solve (1.24) to yield corresponding symplectic gauge fields  $C_a(y)$ . In the end, one may derive the equations of motion for the dynamical gauge

fields  $\hat{A}_\mu(y)$ . We will illustrate later with some examples how this bottom-up approach nicely works.

Note that the symplectic gauge fields in (1.10) are obtained by solving the Darboux transformation (1.3) and they completely determine the gravitational metric (1.26). We want to emphasize that the emergence of gravity originates from the global existence of the one-parameter family of diffeomorphisms describing the local deformation of an initial symplectic structure  $B$  due to the electromagnetic force  $F = dA$ . This essential point can be understood as follows [9]. The symplectic structure  $B$  is a nondegenerate, closed 2-form, i.e.  $dB = 0$ . Therefore the symplectic structure  $B$  defines a bundle isomorphism  $B : TM \rightarrow T^*M$  by  $X \mapsto A = \iota_X B$  where  $\iota_X$  is an interior product with respect to a vector field  $X \in \Gamma(TM)$ . Then the electromagnetic force can be represented by  $F = dA = d\iota_X B = \mathcal{L}_X B$  where the formula (2.8) and  $dB = 0$  were used. This means that the electromagnetic force  $F = dA = \mathcal{L}_X B$  can always be eliminated by a coordinate transformation generated by the vector field  $X$ . (See eq.(23) in [4] for an explicit verification.) This fact vindicates that the emergent gravity reproduces general relativity which also respects diffeomorphism symmetry.

Therefore one can interpret the Darboux transformation (1.3) in symplectic geometry from the viewpoint of emergent gravity described by the metric (1.26) (i.e., in the context of Riemannian geometry). First one can notice that, when fluctuations are turned off, i.e.  $F(X) = 0$ , the symplectic gauge field is given by  $C_a(y) = B_{ab}y^b$  and the corresponding vector fields reduce to  $V_a = \delta_a^\mu \partial_\mu$  and so  $\lambda = 1$  by (2.12). In this case, one can immediately see that the metric (1.26) becomes flat,  $g_{\mu\nu} = \delta_{\mu\nu}$ , and the symplectic volume form reduces to  $\nu = d^4x = \frac{1}{2\text{Pf}B} B \wedge B$  which will be called an asymptotic volume form. Hence it turns out [3, 4, 5] that the flat spacetime is emergent from the vacuum condensate which admits an underlying symplectic structure  $B$  to the vacuum. Now, if one turns on fluctuations, i.e.  $F(X) \neq 0$ , the symplectic gauge field will deviate from the vacuum one and it is given by (1.10). As a result, the metric (1.26) will also deviate from the flat metric, namely,  $V_a : \delta_a^\mu \partial_\mu \rightarrow V_a^\mu(y) \partial_\mu$  and  $g_{\mu\nu} : \delta_{\mu\nu} \rightarrow g_{\mu\nu}(x)$ . But, according to the Darboux theorem or the Moser lemma, one can properly choose a Darboux frame, say on  $U \subset M$  that locally nullifies the fluctuations, and the metric (1.26) on the Darboux chart  $U \subset M$  then locally looks like a flat metric, i.e.  $g_{\mu\nu}|_U = \delta_{\mu\nu}$ . Therefore it would be reasonable to think that a local Darboux chart in symplectic geometry corresponds to a local inertial frame in general relativity.<sup>6</sup> Consequently, if Einstein gravity arises from symplectic gauge fields in the way we have described, the equivalence principle, the most important property in general relativity, might be explained by the Darboux theorem and the Moser lemma in symplectic geometry [3].

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<sup>6</sup>There is a subtle but important difference between the Riemannian and symplectic geometries [9]. Strictly speaking, the equivalence principle in general relativity is a point-wise statement (up to first-order differentials of metric) at a given point  $P$  while the Darboux theorem in symplectic geometry is defined on an entire neighborhood around  $P$ . This is the reason why there exist local invariants, e.g. curvature tensors, in Riemannian geometry but there is no such kind of local invariant in the symplectic geometry. This raises a question how Riemannian geometry is emergent from symplectic geometry though their local geometries are in sharp contrast with each other. A possible resolution was suggested in [9] (see section 2.3). See also [8].

The condition (2.10) says that the vector fields  $\{V_a\}$  are volume-preserving with respect to the volume element  $\nu = \lambda^{-2}\nu_g$ . Suppose that we have chosen a Darboux frame where the ordinary divergence-free condition  $\partial_\mu V_a^\mu = 0$  is obeyed. For such a case, we have the relation

$$\lambda^2 = \det V_a^\mu = \det e_\mu^a = \sqrt{\det g_{\mu\nu}} \quad (2.13)$$

and the symplectic volume  $\nu$  is equal to the asymptotic volume, i.e.,

$$\nu = \frac{1}{2\text{Pf}B} B \wedge B = d^4y. \quad (2.14)$$

Therefore the symplectic volume  $\nu$  remains the same as the asymptotic volume (2.14) even after turning on the fluctuations. Actually this is known as the Liouville theorem in Hamiltonian mechanics, since the vector fields  $\{V_a\}$  in this case are usual Hamiltonian vector fields. But we have remarked in section 1 that the vector fields in a general coordinate system do not always satisfy the condition  $\partial_\mu V_a^\mu = 0$ . In this case the symplectic volume form is not equal to the asymptotic one.

Let us explore gravitational field equations for the metric (1.26) emergent from symplectic gauge fields in (1.10). First note the following relations:

$$\begin{aligned} \{C_a, C_b\}_\theta &= -B_{ab} + \partial_a \hat{A}_b - \partial_b \hat{A}_a + \{\hat{A}_a, \hat{A}_b\}_\theta \\ &\equiv -B_{ab} + \hat{F}_{ab}, \end{aligned} \quad (2.15)$$

$$\begin{aligned} \{C_a, \{C_b, C_c\}_\theta\}_\theta &= \partial_a \hat{F}_{bc} + \{\hat{A}_a, \hat{F}_{bc}\}_\theta \\ &\equiv \hat{D}_a \hat{F}_{bc}. \end{aligned} \quad (2.16)$$

Using the identity (1.14) and the Lie algebra homomorphism (1.20), one can complete the important isomorphism  $\mathfrak{H} : \mathfrak{P}(M) \rightarrow \mathfrak{X}(M)$  between the set of symplectic gauge fields in (1.10) and the vector fields in (1.21) which is represented by [9, 30]

$$X_{\hat{F}_{ab}} = X_{\{C_a, C_b\}_\theta} = [V_a, V_b], \quad (2.17)$$

$$X_{\hat{D}_a \hat{F}_{bc}} = X_{\{C_a, \{C_b, C_c\}_\theta\}_\theta} = [V_a, [V_b, V_c]]. \quad (2.18)$$

Adopting the same method as (1.24), one can derive from (2.17) the following relation

$$\{\hat{F}_{ab}, x^\mu\}_\theta(y) = -g_{ab}{}^c V_c^\mu(x) \quad (2.19)$$

where  $g_{ab}{}^c$  are structure coefficients in (2.11). In the bottom-up approach, the right-hand side of (2.19) is determined by a given metric and so, in principle, one can solve it to determine the field strength  $\hat{F}_{ab}(y)$  of symplectic gauge fields.

A remarkable point is that the electromagnetism on a symplectic manifold  $(M, B)$  is completely described by the Poisson algebra  $\mathfrak{P}(M) = (C^\infty(M), \{-, -\}_\theta)$  [4]. For example, the action is given by

$$S = \frac{1}{4g_{YM}^2} \int d^4x \{C_a, C_b\}_\theta^2. \quad (2.20)$$

The identity (2.18) provides us a direct map [3, 5, 9, 30] to connect the equations of motion of symplectic gauge fields derived from the action (2.20) to gravitational field equations for

the emergent metric (1.26):

$$\widehat{D}_a \widehat{F}_{bc} + \text{cyclic} = 0 \quad \Leftrightarrow \quad [V_a, [V_b, V_c]] + \text{cyclic} = 0, \quad (2.21)$$

$$\widehat{D}_b \widehat{F}_{ab} = 0 \quad \Leftrightarrow \quad [V_b, [V_a, V_b]] = 0. \quad (2.22)$$

It can be shown (even in any  $2n$ -dimensions) [9, 5] that the right-hand side of (2.21) is precisely equivalent to the first Bianchi identity of Riemann curvature tensors, i.e.,

$$[V_a, [V_b, V_c]] + \text{cyclic} = 0 \quad \Leftrightarrow \quad R_{[abc]d} = 0, \quad (2.23)$$

where  $[abc]$  denotes the cyclic permutation of indices. The equations of motion (2.22) leads to a cryptic result for Ricci tensors [9, 5]

$$R_{ab} = -\frac{1}{\lambda^2} \left[ g_d^{(+i)} g_d^{(-j)} \left( \eta_{ac}^i \bar{\eta}_{bc}^j + \eta_{bc}^i \bar{\eta}_{ac}^j \right) - g_c^{(+i)} g_d^{(-j)} \left( \eta_{ac}^i \bar{\eta}_{bd}^j + \eta_{bc}^i \bar{\eta}_{ad}^j \right) \right] \quad (2.24)$$

where  $\eta_{ab}^i$  and  $\bar{\eta}_{ab}^i$  are self-dual and anti-self-dual 't Hooft symbols. To get the result (2.24), we have defined the canonical decomposition of the structure equation (2.11)

$$g_{abc} = g_c^{(+i)} \eta_{ab}^i + g_c^{(-i)} \bar{\eta}_{ab}^i. \quad (2.25)$$

A notable point is that the right-hand side of (2.24) consists of purely interaction terms between self-dual and anti-self-dual parts in (2.25) which is the feature withheld by matter fields only [31, 32]. A gravitational instanton which is a Ricci-flat, Kähler manifold can be understood as either  $g_c^{(-i)} = 0$  (self-dual) or  $g_c^{(+i)} = 0$  (anti-self-dual) in terms of (2.25) and so  $R_{ab} = 0$  in (2.24). Hence, the result (2.24) is consistent with the Ricci-flatness of gravitational instantons. However a unique property of (2.24) is to contain a nontrivial trace contribution, i.e., a nonzero Ricci scalar, due to the second part which is non-existent in Einstein gravity as was recently shown in [32]. The content of the energy-momentum tensor defined by the right-hand side of (2.24) becomes manifest by decomposing it into two parts, denoted by  $8\pi GT_{ab}^{(M)}$  and  $8\pi GT_{ab}^{(L)}$ , respectively [9, 5]:

$$8\pi GT_{ab}^{(M)} = -\frac{1}{\lambda^2} \left( g_{acd} g_{bcd} - \frac{1}{4} \delta_{ab} g_{cde} g_{cde} \right), \quad (2.26)$$

$$8\pi GT_{ab}^{(L)} = \frac{1}{2\lambda^2} \left( \rho_a \rho_b - \Psi_a \Psi_b - \frac{1}{2} \delta_{ab} (\rho_c^2 - \Psi_c^2) \right), \quad (2.27)$$

where

$$\rho_a \equiv g_{bab}, \quad \Psi_a \equiv -\frac{1}{2} \varepsilon^{abcd} g_{bcd}. \quad (2.28)$$

The first energy-momentum tensor (2.26) is traceless, i.e.  $8\pi GT_{aa}^{(M)} = 0$ , which is a consequence of the identity  $\eta_{ab}^i \bar{\eta}_{ab}^j = 0$  when applied to the first part of (2.24). The Ricci scalar  $R \equiv R_{aa}$  can be calculated by (2.27) to yield

$$R = \frac{1}{2\lambda^2} \left( \rho_a^2 - \Psi_a^2 \right). \quad (2.29)$$

The equation (2.29) immediately leads to the conclusion that a four-manifold emergent from pure symplectic gauge fields (without source terms) can have a vanishing Ricci scalar if and only if

$$\rho_a = \pm \Psi_a \quad (2.30)$$

that is similar to the self-duality equation. When the relation (2.30) is obeyed, the second energy-momentum tensor  $8\pi GT_{ab}^{(L)}$  identically vanishes. In section 4 we will show that the LeBrun metric [24] satisfies the relation (2.30) and so it can arise in emergent gravity from pure symplectic gauge fields.

It would be worthwhile to remark that a four-manifold with a vanishing Ricci scalar cannot be realized as a vacuum solution of Einstein gravity without matter fields. Indeed the Einstein's equation can be written as  $R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R = 8\pi G\tilde{T}_{\mu\nu}$  where  $\tilde{T}_{\mu\nu} = T_{\mu\nu} - \frac{1}{4}g_{\mu\nu}T$  is a traceless energy-momentum tensor which annuls a possible cosmological constant. For a scalar-flat four-manifold, the Einstein equations reduce to  $R_{\mu\nu} = 8\pi G\tilde{T}_{\mu\nu}$ . This means, a scalar-flat manifold can only arise from a traceless (conformal) matter source. This condition can be realized in Einstein-Yang-Mills systems in four-dimensions. For instance, the LeBrun metric is a solution of Euclidean Einstein-Maxwell theory [33, 34, 35]. There are some reasons that the energy-momentum tensor (2.26) can be mapped to that of the usual Maxwell theory in commutative spacetime. Indeed it was argued in [9] that it can be done by reversing the map (1.9). Hence, the emergent gravity shows that such a scalar-flat four-manifold can emerge from pure Maxwell theory on a symplectic manifold  $(M, B)$ .

### 3. Scalar-flat Kähler metrics

LeBrun found in [24] the explicit local form of all Euclidean, four-dimensional Kähler metrics that have a  $U(1)$  isometry and a vanishing Ricci scalar. It is then shown in [33] that these metrics are necessarily solutions of Einstein-Maxwell theory whose electromagnetic field is related to the Kähler form. The LeBrun metric takes the form

$$ds^2 = w^{-1}(d\tau + A)^2 + w(e^u(dx^2 + dy^2) + dz^2) \quad (3.1)$$

where  $w > 0$  and  $u$  are smooth real-valued functions on an open set  $U \subset \mathbb{R}^3$  which satisfy the  $su(\infty)$  Toda equation and its linearized form:

$$\partial_x^2 u + \partial_y^2 u + \partial_z^2(e^u) = 0, \quad (3.2)$$

$$\partial_x^2 w + \partial_y^2 w + \partial_z^2(e^u w) = 0. \quad (3.3)$$

The one-form,  $A$ , obeys

$$dA = \partial_x w dy \wedge dz + \partial_y w dz \wedge dx + \partial_z(e^u w) dx \wedge dy \quad (3.4)$$

and the closedness of  $dA$ , i.e.  $d^2 A = 0$ , is equivalent to the equation (3.3). The Kähler form is given by

$$\Omega = (d\tau + A) \wedge dz - we^u dx \wedge dy \quad (3.5)$$

and the metric (3.1) is Kähler, i.e.,  $d\Omega = 0$ .

The LeBrun metric (3.1) is defined by two functions,  $u(\mathbf{x})$  and  $w(\mathbf{x})$ , on an open set  $U \subset \mathbb{R}^3 \ni \mathbf{x}$ , satisfying (3.2) and (3.3), respectively. But one may consider the function  $w(\mathbf{x})$  as a linear perturbation of  $u(\mathbf{x})$  from a Toda point  $u_t(\mathbf{x})$  which satisfies the  $su(\infty)$  Toda equation (3.2). Then the equation (3.3) implies that a linear deviation of the function

$u(\mathbf{x})$  from a Toda point  $u_t(\mathbf{x})$ , that is  $u(\mathbf{x}) = u_t(\mathbf{x}) + w(\mathbf{x})$  and so  $e^{u(\mathbf{x})} \approx e^{u_t(\mathbf{x})} + e^{u_t(\mathbf{x})} w(\mathbf{x})$ , is still a solution of the  $su(\infty)$  Toda equation (3.2). For example, if the Toda point is  $u_t = 0$ , we get the Gibbons-Hawking metric and, if  $w(\mathbf{x})$  is generated by a  $z$ -translation from a Toda point  $u_t$  with  $A_3 = 0$ , i.e.  $u(z + \epsilon) \approx u_t(z) + \epsilon \partial_z u(z) := u_t(\mathbf{x}) + w(\mathbf{x})$ , the metric (3.1) gives rise to the real heaven. Indeed the Gibbons-Hawking metric [19] takes the form

$$ds^2 = w^{-1}(d\tau + A)^2 + w(dx^2 + dy^2 + dz^2) \quad (3.6)$$

and  $w(\mathbf{x})$  is a harmonic function on  $\mathbb{R}^3$ . The  $U(1)$  gauge field  $A$  satisfies the self-duality equation

$$\nabla \times \mathbf{A} = \nabla w \quad (3.7)$$

which is precisely (3.4) and is consistent with the harmonic equation (3.3) (with  $u = 0$ ). And the explicit form of the real heaven metric [25, 26] is given by

$$ds^2 = (\partial_z u)^{-1}(d\tau + a)^2 + \partial_z u(e^u(dx^2 + dy^2) + dz^2) \quad (3.8)$$

where  $a = \partial_y u dx - \partial_x u dy$ . Since the function  $u(\mathbf{x})$  obeys the continual Toda equation (3.2), it is easy to see that the  $U(1)$  field strength  $F = da$  is equal to (3.4).

Another interesting Toda point is given by

$$u = \log 2z \quad (3.9)$$

which is definitely a solution of the equation (3.2). In this case, the so-called LeBrun-Burns metric [24] can be written in the form

$$ds^2 = \zeta^2 \left( V^{-1}(d\tau + A)^2 + V \left( \frac{dx^2 + dy^2 + d\zeta^2}{\zeta^2} \right) \right) \quad (3.10)$$

by introducing a new coordinate  $\zeta \equiv \sqrt{2z}$  and a new potential  $V \equiv we^u = \zeta^2 w$ . Note that the three-dimensional metric is the standard constant-curvature metric on the hyperbolic plane  $\mathbb{H}_3$ :

$$ds_{\mathbb{H}_3}^2 = \frac{dx^2 + dy^2 + d\zeta^2}{\zeta^2}. \quad (3.11)$$

Then the equations (3.3) and (3.4) imply that  $V$  is a harmonic function on the hyperbolic plane and  $A$  satisfies an appropriate self-duality equation on  $\mathbb{H}_3$  [34, 35]:

$$\nabla_{\mathbb{H}_3}^2 V = 0, \quad dA = *\mathbb{H}_3 dV. \quad (3.12)$$

Therefore the LeBrun-Burns metric (3.10) provides a hyperbolic analogue of the Gibbons-Hawking metric (3.6) although it is not a hyper-Kähler manifold but just a Kähler manifold with vanishing scalar curvature.

It is convenient to introduce coframes for the LeBrun metric (3.1)

$$e^1 = w^{\frac{1}{2}} e^{\frac{u}{2}} dx, \quad e^2 = w^{\frac{1}{2}} e^{\frac{u}{2}} dy, \quad e^3 = w^{\frac{1}{2}} dz, \quad e^4 = w^{-\frac{1}{2}}(d\tau + A), \quad (3.13)$$

and frames

$$\begin{aligned} E_1 &= w^{-\frac{1}{2}} e^{-\frac{u}{2}} \left( \frac{\partial}{\partial x} - A_1 \frac{\partial}{\partial \tau} \right), & E_2 &= w^{-\frac{1}{2}} e^{-\frac{u}{2}} \left( \frac{\partial}{\partial y} - A_2 \frac{\partial}{\partial \tau} \right), \\ E_3 &= w^{-\frac{1}{2}} \left( \frac{\partial}{\partial z} - A_3 \frac{\partial}{\partial \tau} \right), & E_4 &= w^{\frac{1}{2}} \frac{\partial}{\partial \tau}. \end{aligned} \quad (3.14)$$

The Poisson bivector  $\Theta \equiv \Omega^{-1} \in \Gamma(\wedge^2 TM)$  determined by the Kähler form  $\Omega = -(e^1 \wedge e^2 + e^3 \wedge e^4)$  takes the form

$$\Theta = \frac{1}{2} \Theta^{\mu\nu}(x) \frac{\partial}{\partial x^\mu} \wedge \frac{\partial}{\partial x^\nu} = E_1 \wedge E_2 + E_3 \wedge E_4. \quad (3.15)$$

For our later purpose, we present explicit forms of the Kähler form and the Poisson tensor:

$$\Omega_{\mu\nu} = \begin{pmatrix} 0 & -we^u & A_1 & 0 \\ we^u & 0 & A_2 & 0 \\ -A_1 & -A_2 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad (3.16)$$

$$\Theta^{\mu\nu} = \begin{pmatrix} 0 & w^{-1}e^{-u} & 0 & -w^{-1}e^{-u}A_2 \\ -w^{-1}e^{-u} & 0 & 0 & w^{-1}e^{-u}A_1 \\ 0 & 0 & 0 & 1 \\ w^{-1}e^{-u}A_2 & -w^{-1}e^{-u}A_1 & -1 & 0 \end{pmatrix}. \quad (3.17)$$

It was shown in [33, 34] that the LeBrun metric is a solution of Euclidean Einstein-Maxwell equations

$$R_{\mu\nu} = \frac{1}{2} \left( \mathcal{F}_{\mu\rho} \mathcal{F}_\nu{}^\rho - \frac{1}{4} g_{\mu\nu} \mathcal{F}_{\rho\sigma} \mathcal{F}^{\rho\sigma} \right). \quad (3.18)$$

The Maxwell field strength  $\mathcal{F} \equiv F + \Omega$  is given by

$$\mathcal{F} = -\frac{1}{2} \sum_{i=1}^3 \partial_i \left( \frac{\partial_z u}{w} \right) \Omega_-^{(i)} + \Omega \quad (3.19)$$

where  $\Omega$  is the Kähler form (3.5) and  $\Omega_-^{(i)}$  are anti-self-dual forms defined by

$$\begin{aligned} \Omega_-^{(1)} &= e^{-\frac{u}{2}} (e^2 \wedge e^3 - e^1 \wedge e^4), \\ \Omega_-^{(2)} &= e^{-\frac{u}{2}} (e^3 \wedge e^1 - e^2 \wedge e^4), \\ \Omega_-^{(3)} &= e^1 \wedge e^2 - e^3 \wedge e^4. \end{aligned} \quad (3.20)$$

Our convention for the anti-self-dual forms in (3.20) is actually the orientation flip of the self-dual forms in [34] because the Riemannian volume form in our case is given by  $\nu_g = \Upsilon dx \wedge dy \wedge dz \wedge d\tau$  with  $\Upsilon = we^u$  while the volume form in [34] is given by  $\nu_g = \Upsilon d\tau \wedge dx \wedge dy \wedge dz$ . The two-form  $F = dC$  in (3.19) has a vector potential given by

$$C = \frac{1}{2} \left( \left( \frac{\partial_z u}{w} \right) (d\tau + A) - \partial_y u dx + \partial_x u dy \right). \quad (3.21)$$

Note that  $F = dC$  identically vanishes for self-dual manifolds such as the Gibbons-Hawking metric and the real heaven. In this case, the Maxwell field strength  $\mathcal{F} = \Omega$  is simply given by the self-dual Kähler form (3.5) and so the energy-momentum tensor in (3.18) is identically zero to yield Ricci-flat manifolds.

## 4. U(1) gauge fields from scalar-flat Kähler metrics

Now we will explore symplectic gauge fields defined by the map (1.24). First let us start with warmup examples - gravitational instantons [36, 37] (hyper-Kähler manifolds) in (3.6) and (3.8) and then consider the general case described by the LeBrun metric (3.1).

### 4.1 Gibbons-Hawking metric

For the Gibbons-Hawking metric (3.6), it is easy to solve (2.7) using the inverse vierbeins in (3.14) (with  $u = 0$ ) to determine  $\lambda$  given by

$$\lambda = w^{\frac{1}{2}}. \quad (4.1)$$

Then the relation (1.25) determines the vector fields  $V_a = \lambda E_a \in \Gamma(TM)$  to be [23]

$$\begin{aligned} V_1 &= \frac{\partial}{\partial x} - A_1 \frac{\partial}{\partial \tau}, & V_2 &= \frac{\partial}{\partial y} - A_2 \frac{\partial}{\partial \tau}, \\ V_3 &= \frac{\partial}{\partial z} - A_3 \frac{\partial}{\partial \tau}, & V_4 &= w \frac{\partial}{\partial \tau}. \end{aligned} \quad (4.2)$$

It is easy to check using (3.7) that the vector fields (4.2) satisfy the anti-self-duality equation [21, 22, 23]

$$[V_a, V_b] = -\frac{1}{2} \varepsilon_{ab}{}^{cd} [V_c, V_d]. \quad (4.3)$$

Moreover the vector fields (4.2) are divergence-free, i.e.,  $\partial_\mu V_a^\mu(x) = 0$ .

From the vector fields in (4.2), we can determine the Poisson system for symplectic gauge fields to obey:

$$V_a(x^i) = \{C_a, x^i\}_\theta(y) = \delta_a^i, \quad (4.4)$$

$$V_a(\tau) = \{C_a, \tau\}_\theta(y) \equiv Y_a(x) = (-A_i, w)(x). \quad (4.5)$$

One might try to directly solve the Poisson system (4.4) and (4.5) of the partial differential equations to determine the symplectic gauge fields  $C_a(y)$ . However it is not possible unless  $x^\mu(y)$  are explicitly known. One way to avoid this is by going to a particular frame using the symmetry (1.14) where the underlying symplectic structure is defined by the Kähler form (3.5) itself. In this Kähler frame, the vector fields are given by

$$V_a^\mu(x) = \{C_a(x), x^\mu\}_\Theta = -\Theta^{\mu\nu}(x) \frac{\partial C_a(x)}{\partial x^\nu} \quad (4.6)$$

using the Poisson tensor  $\Theta^{\mu\nu}$  in (3.17) (with  $u = 0$ ). It will be useful to have the explicit expression for  $J_{ab} \equiv \frac{\partial C_a(x)}{\partial x^b}$ :

$$J_{ab} = \begin{pmatrix} 0 & -w & 0 & 0 \\ w & 0 & 0 & 0 \\ -A_1 & -A_2 & -A_3 & -1 \\ 0 & 0 & w & 0 \end{pmatrix} \quad (4.7)$$

where  $a$  is the row index and  $b$  is the column index. Then it is easy to see that  $J_{ab} - J_{ba} = \Omega_{ab} - w\eta_{ab}^3$  where  $\Omega_{ab}$  are components of the Kähler form in (3.5) and  $\eta_{ab}^3$  is the self-dual 't Hooft symbol. It implies that  $J_{ab}dx^a \wedge dx^b + \frac{w}{2}\eta_{ab}^3 dx^a \wedge dx^b = \Omega$  is a closed two-form.



In order to get a better handle over the differential equation (4.7), it would be worthwhile to appreciate that the symplectic gauge fields  $C_a(x)$  in (1.10) are non-local functions in general, effectively describing the dynamics of dipole-like objects. Actually one should not insist that  $C_a(x)$  are local functions because  $\hat{A}_a(x)$  in (1.10) corresponds to a leading approximation of NC U(1) gauge fields up to  $\mathcal{O}(\theta)$  whose physical excitations are described by NC dipoles—weakly interacting, nonlocal objects [38, 39]. In order to clarify this point, let us introduce an open Wilson line [40, 41, 42] which plays a crucial role in the Seiberg-Witten map [14]. First consider a path  $P$  parameterized by  $\zeta^\mu(\sigma) = \theta^{\mu\nu} k_\nu \sigma$  with  $0 \leq \sigma \leq 1$  and define a curve by

$$x^\mu(\sigma) = x_0^\mu + \zeta^\mu(\sigma) \quad (4.8)$$

with  $x^\mu(\sigma = 0) \equiv x_0^\mu$  and  $x^\mu(\sigma = 1) \equiv x^\mu$ . If one considers the following symplectic gauge fields defined by (see the second equation of (54) in [43])

$$\int_0^1 d\sigma \frac{dx^\lambda(\sigma)}{d\sigma} J_{\mu\lambda}(x(\sigma)) = C_\mu(x) - C_\mu(x_0), \quad (4.9)$$

they obey (4.7). Here we are applying the following formula

$$\frac{\partial}{\partial x^\mu} \int_0^1 d\sigma \frac{dx^\lambda(\sigma)}{d\sigma} K(x(\sigma)) = \delta_\mu^\lambda K(x) \quad (4.10)$$

for some differentiable function  $K(x)$ . Note that the dipole field in (4.9) is an extended object with size  $|\zeta| = |x - x_0|$  but the vector fields  $V_a$  become local as usual although symplectic gauge fields could be non-local.<sup>7</sup>

It seems quite nontrivial to solve (4.9) for general (multi-centered) Gibbons-Hawking metric (even for the simplest Eguchi-Hanson metric demands for a separate work [44]). Rather we will determine the equations of motion that the symplectic gauge fields must satisfy. In this respect, we can apply the Lie algebra homomorphism (2.17) to the anti-self-duality equation (4.3) to show that the U(1) field strength is anti-self-dual, i.e.,

$$\hat{F}_{ab} = -\frac{1}{2} \varepsilon_{ab}{}^{cd} \hat{F}_{cd}. \quad (4.11)$$

This is consistent with the result [45] in the top-down approach that the Gibbons-Hawking metric arises from symplectic U(1) instantons. Furthermore, since the vector fields in (4.2) arise from a specific solution, the symplectic gauge fields for the Gibbons-Hawking metric (3.6) are further constrained. In order to discuss this aspect, it is convenient to introduce the Jacobiator defined by

$$J(f, g, h) \equiv \{\{f, g\}_\theta, h\}_\theta + \{\{g, h\}_\theta, f\}_\theta + \{\{h, f\}_\theta, g\}_\theta \quad (4.12)$$

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<sup>7</sup>It should be noticed that the symplectomorphism,  $x^\mu = y^\mu + \{y^\mu, \phi\}_\theta$ , is in fact equivalent to U(1) gauge transformation [8]. In particular the symplectomorphism with the gauge parameter  $\phi = k_\mu y^\mu$  generates a translation  $x^\mu = y^\mu + \zeta^\mu$  with  $\zeta^\mu = \theta^{\mu\nu} k_\nu$ . Hence the two points  $x^\mu$  and  $y^\mu$  are on the same gauge orbit, i.e.  $x^\mu \sim y^\mu$ . Therefore the dipole field (4.9) actually behaves like a closed loop in “physical phase space.” This closed string picture for dipole fields was further elaborated in [39].

for  $f, g, h \in C^\infty(M)$ . The Jacobi identity  $J(C_a, C_b, x^i) = 0$  then leads to the result  $\{\widehat{F}_{ab}, x^i\}_\theta = 0$ . Combining it with the Lie algebra relation (2.19) yields the condition

$$\{\widehat{F}_{ab}, x^i\}_\theta = -g_{ab}{}^i = 0. \quad (4.13)$$

It is easy to check that the vector fields in (4.2) indeed satisfy (4.13) and nonzero components are given by

$$g_{ab}{}^4 = -\frac{1}{w}(\delta_a^i \partial_i Y_b - \delta_b^i \partial_i Y_a). \quad (4.14)$$

Similarly the Jacobi identity  $J(C_a, C_b, \tau) = 0$  leads to the relation

$$\{\widehat{F}_{ab}, \tau\}_\theta = -g_{ab}{}^4 w = \{C_a, Y_b\}_\theta - \{C_b, Y_a\}_\theta \quad (4.15)$$

where we imposed the condition (4.13). Of course, the above equation must be anti-self-dual with respect to  $(a, b)$  index pair.

Using the same strategy as (4.6), one can represent (4.13) and (4.15) in the Kähler frame (3.17) and the result can be written as

$$\frac{\partial \widehat{F}_{ab}}{\partial x} = \frac{\partial \widehat{F}_{ab}}{\partial y} = \frac{\partial \widehat{F}_{ab}}{\partial \tau} = 0, \quad (4.16)$$

$$\frac{\partial \widehat{F}_{ab}}{\partial z}(x) = -g_{ab}{}^4 w(x) = V_a(Y_b)(x) - V_b(Y_a)(x) \quad (4.17)$$

where

$$g_{ij}{}^4 = \varepsilon_{ijk} \partial_k \log w, \quad g_{4i}{}^4 = \partial_i \log w. \quad (4.18)$$

The above equations imply that, if we solve the self-duality equation (4.11) with the U(1) field strength given by

$$\widehat{F}_{ab}(x) = B_{ab} + \Theta^{\mu\nu}(x) \frac{\partial C_a(x)}{\partial x^\mu} \frac{\partial C_b(x)}{\partial x^\nu}, \quad (4.19)$$

then the dipole field for the U(1) field strength extends along  $z$ -direction only (according to the formula (4.10)). It will be interesting to explicitly solve (4.17) using the Gibbons-Hawking metric (3.6). We want to postpone this project to future works which will be initiated in [44]. Anyway the bottom-up approach again proves the equivalence [46, 47, 45] between gravitational instantons and symplectic U(1) instantons, rigorously established from the top-down approach [9, 3, 5].

## 4.2 Real heaven

The real heaven metric (3.8) can be analyzed precisely in the same way as the Gibbons-Hawking case except for the fact that a frame rotation is necessary to solve (2.7). Let us take a particular SO(4) rotation

$$\begin{pmatrix} E'_3 \\ E'_4 \end{pmatrix} = \begin{pmatrix} \cos \frac{\tau}{2} & -\sin \frac{\tau}{2} \\ \sin \frac{\tau}{2} & \cos \frac{\tau}{2} \end{pmatrix} \begin{pmatrix} E_3 \\ E_4 \end{pmatrix}, \quad (4.20)$$

leaving (1-2)-plane unchanged. In this rotated frame, it is easy to solve (2.7) using the inverse vierbeins in (3.14) (with  $w = \partial_z u$  and  $A_3 = 0$ ) to determine  $\lambda$  given by

$$\lambda = w^{\frac{1}{2}} e^{\frac{u}{2}}. \quad (4.21)$$

The vector fields  $V_a \in \Gamma(TM)$  in the rotated frame are determined by the relation (1.25) as (after dropping the prime) [48]

$$\begin{aligned} V_1 &= \frac{\partial}{\partial x} - a_1 \frac{\partial}{\partial \tau}, & V_2 &= \frac{\partial}{\partial y} - a_2 \frac{\partial}{\partial \tau}, \\ V_3 &= e^{\frac{u}{2}} \left( \cos \frac{\tau}{2} \frac{\partial}{\partial z} - \partial_z u \sin \frac{\tau}{2} \frac{\partial}{\partial \tau} \right), & V_4 &= e^{\frac{u}{2}} \left( \sin \frac{\tau}{2} \frac{\partial}{\partial z} + \partial_z u \cos \frac{\tau}{2} \frac{\partial}{\partial \tau} \right) \end{aligned} \quad (4.22)$$

where  $a_i = \varepsilon_{ij} \partial_j u$  ( $i, j = 1, 2$ ). The above vector fields together with (3.2) and (3.4) immediately show that they also satisfy the anti-self-duality equation [21, 22, 23]

$$[V_a, V_b] = -\frac{1}{2} \varepsilon_{ab}{}^{cd} [V_c, V_d]. \quad (4.23)$$

Furthermore the vector fields (4.22) are also divergence-free, i.e.,  $\partial_\mu V_a^\mu(x) = 0$ . Thus the U(1) field strength derived from the real heaven metric (3.8) must be anti-self-dual, i.e.,

$$\widehat{F}_{ab} = -\frac{1}{2} \varepsilon_{ab}{}^{cd} \widehat{F}_{cd}. \quad (4.24)$$

This is also consistent with the top-down approach as was shown in [45].

For convenience let us explicitly rewrite the components of the vector fields (4.22) defined by (4.6)

$$V_a^\mu(x) = \begin{pmatrix} 1 & 0 & 0 & -a_1 \\ 0 & 1 & 0 & -a_2 \\ 0 & 0 & e^{\frac{u}{2}} \cos \frac{\tau}{2} & -e^{\frac{u}{2}} \partial_z u \sin \frac{\tau}{2} \\ 0 & 0 & e^{\frac{u}{2}} \sin \frac{\tau}{2} & e^{\frac{u}{2}} \partial_z u \cos \frac{\tau}{2} \end{pmatrix}. \quad (4.25)$$

The corresponding matrix  $J_{ab} = \frac{\partial C_a}{\partial x^b} = V_a^\lambda \Omega_{\lambda b}$  for the real heaven metric is given by

$$J_{ab}(x) = \begin{pmatrix} 0 & -we^u & 0 & 0 \\ we^u & 0 & 0 & 0 \\ -a_1 e^{\frac{u}{2}} \cos \frac{\tau}{2} & -a_2 e^{\frac{u}{2}} \cos \frac{\tau}{2} & -e^{\frac{u}{2}} \partial_z u \sin \frac{\tau}{2} & -e^{\frac{u}{2}} \cos \frac{\tau}{2} \\ -a_1 e^{\frac{u}{2}} \sin \frac{\tau}{2} & -a_2 e^{\frac{u}{2}} \sin \frac{\tau}{2} & e^{\frac{u}{2}} \partial_z u \cos \frac{\tau}{2} & -e^{\frac{u}{2}} \sin \frac{\tau}{2} \end{pmatrix}. \quad (4.26)$$

The symplectic gauge fields  $C_a(x)$  for the real heaven can also be solved by introducing a dipole field similar to (4.9). However this case is more complicated than (4.9) because the path  $P$  for the open Wilson line has to be placed in four-dimensional space parameterized by  $(\tau, \mathbf{x})$  while for the Gibbons-Hawking case it was enough to span three-dimensional space  $\mathbb{R}^3$  parameterized by  $\mathbf{x}$ . Moreover the Jacobi identity  $J(C_a, C_b, x^i) = \{\widehat{F}_{ab}(x), x^i\}_\Theta = 0$  for  $i = 1, 2$  implies only the condition

$$(\partial_v - a_v \partial_\tau) \widehat{F}_{ab}(x) = 0 \quad (4.27)$$

where  $v = \frac{1}{2}(x + iy)$ ,  $\partial_v = \partial_x - i\partial_y$  and  $a_v = a_1 - ia_2 = i\partial_v u$ .

But we may simplify the problem as was noticed in [49]. The starting point is to observe that the system of vector fields in (4.22) can be regarded as Hamiltonian vector fields on  $\mathbb{R}^2 \times \Sigma$  where  $(\Sigma, \omega)$  is a two-dimensional symplectic manifold. So let us represent them as

$$V_1 = \frac{\partial}{\partial x} + W_{\psi_1}, \quad V_2 = \frac{\partial}{\partial y} + W_{\psi_2}, \quad V_3 = W_{\psi_3}, \quad V_4 = W_{\psi_4} \quad (4.28)$$

where  $W_{\psi_a} \equiv \psi_a^\alpha(x) \partial_\alpha$  ( $\alpha = 1, 2 \in (z, \tau)$ ) are Hamiltonian vector fields on  $\Sigma$  associated with some functions  $\psi_a \in C^\infty(\mathbb{R}^2 \times \Sigma)$ . Then the anti-self-duality equation (4.23) reads as [49]

$$\begin{aligned} \frac{\partial \psi_2}{\partial x} - \frac{\partial \psi_1}{\partial y} + \{\psi_1, \psi_2\} + \{\psi_3, \psi_4\} &= 0, \\ \frac{\partial \psi_3}{\partial x} - \frac{\partial \psi_4}{\partial y} + \{\psi_1, \psi_3\} - \{\psi_2, \psi_4\} &= 0, \\ \frac{\partial \psi_4}{\partial x} + \frac{\partial \psi_3}{\partial y} + \{\psi_1, \psi_4\} + \{\psi_2, \psi_3\} &= 0, \end{aligned} \quad (4.29)$$

where  $\{\psi_a, \psi_b\} = \partial_z \psi_a \partial_\tau \psi_b - \partial_\tau \psi_a \partial_z \psi_b$  denotes the Poisson bracket on  $\Sigma$ . If we wish,  $\Sigma$  can be taken as a Riemann surface of genus  $g$ .

Now we note that the vector fields in (4.28) are precisely the same as those arising from four-dimensional noncommutative  $U(1)$  gauge theory on  $\mathbb{R}_C^2 \times \mathbb{R}_{NC}^2$  which is mapped to two-dimensional  $U(N \rightarrow \infty)$  gauge theory with two adjoint scalar fields (see eq.(3.30) in [9]). From the viewpoint of  $U(N)$  gauge theory, we make the following identification:

$$\psi_i = \hat{a}_i, \quad \psi_3 + i\psi_4 = \Phi, \quad \psi_3 - i\psi_4 = \Phi^\dagger \quad (4.30)$$

where  $\hat{a}_i$  ( $i = 1, 2$ ) and  $\Phi$  are two-dimensional  $U(N \rightarrow \infty)$  gauge fields and a complex adjoint scalar field on  $\mathbb{R}^2$  or equivalently four-dimensional symplectic  $U(1)$  gauge fields on  $\mathbb{R}^2 \times \Sigma$ . Using the notation of (4.30), the anti-self-duality equation (4.29) can be written as (see eq.(4.1) in [50])

$$\hat{F}_{12} = \frac{i}{2} \{\Phi^\dagger, \Phi\}, \quad D_{\bar{v}} \Phi = 0 \quad (4.31)$$

where  $\hat{F}_{12} = \partial_1 \hat{a}_2 - \partial_2 \hat{a}_1 + \{\hat{a}_1, \hat{a}_2\}$  and  $D_{\bar{v}} = D_x + iD_y$ . It is remarkable that the self-dual system (4.23) for the real heaven metric reduces to the BPS equations (4.31) with gauge group  $G = SDiff(\Sigma)$  – area preserving diffeomorphisms on a Riemann surface  $\Sigma$ , for example, or  $U(N \rightarrow \infty)$  after the quantization of  $(\Sigma, \omega)$ . Furthermore it was shown in [50] that the BPS equations (4.31) can be recast into the equation of motion derived from the two-dimensional  $U(N)$  chiral model governed by the action

$$S = \frac{1}{2} \int d^2 z \text{Tr} \partial_\mu h^{-1} \partial_\nu h \delta^{\mu\nu} \quad (4.32)$$

where a group element  $h(z)$  defines a map from  $\mathbb{R}^2$  to  $GL(N, \mathbb{C})$  group, which is contractible to  $U(N) \subset GL(N, \mathbb{C})$ . It has been known [51, 52, 53] that the chiral model (4.32) in the  $N \rightarrow \infty$  limit describes a self-dual spacetime whose equations of motion take the Plebański form of self-dual Einstein equations [54].

Finally it is not difficult to solve the coupled equations (4.29) to determine the symplectic gauge fields in (4.30) and the result is already known thanks to [49]:

$$\hat{a}_1(x) = - \int^z \frac{\partial u}{\partial y} dz, \quad \hat{a}_2(x) = \int^z \frac{\partial u}{\partial x} dz, \quad \Phi(x) = 2e^{\frac{u+i\tau}{2}}. \quad (4.33)$$

Therefore we have got the solution (4.33) of the BPS equations (4.31) based on the bottom-up approach.<sup>8</sup>

### 4.3 LeBrun metric

As was pointed out in section 3, the LeBrun metric (3.1) is a solution of the Einstein-Maxwell equation. Therefore it is nontrivial to solve (2.7) to determine  $\lambda$ . Hence we will show some details of our calculation. For the given frame (3.14), it is also necessary to take a frame rotation like (4.20) but with a modified form

$$\begin{pmatrix} E'_3 \\ E'_4 \end{pmatrix} = \begin{pmatrix} \cos \frac{\phi}{2} & -\sin \frac{\phi}{2} \\ \sin \frac{\phi}{2} & \cos \frac{\phi}{2} \end{pmatrix} \begin{pmatrix} E_3 \\ E_4 \end{pmatrix}, \quad (4.34)$$

where the angle variable  $\phi$  along the U(1) fiber is defined by

$$\phi \equiv \tau + \int^z A_3(\mathbf{x}) dz. \quad (4.35)$$

We still keep (1-2)-plane unchanged. For  $a = 1, 2$ , it is possible to solve (2.7) with  $\lambda = w^{\frac{1}{2}} e^{\frac{u}{2}}$ . But, for  $a = 3, 4$ , there is an extra term with the final result:

$$\begin{aligned} \nabla \cdot E'_3 &= \frac{w^{-\frac{1}{2}}}{2} \cos \frac{\phi}{2} \left( \partial_z \log w e^u - (w - \partial_z u) \right), \\ \nabla \cdot E'_4 &= \frac{w^{-\frac{1}{2}}}{2} \sin \frac{\phi}{2} \left( \partial_z \log w e^u - (w - \partial_z u) \right). \end{aligned} \quad (4.36)$$

Thus, in order to cancel the extra term, it is required to choose  $\lambda$  properly without affecting the result for  $a = 1, 2$ . It turns out that it can be done by introducing a dipole-like object given by

$$\begin{aligned} \lambda &= \exp \left( -\mathbf{k} \cdot \frac{1}{2} \int_0^1 d\sigma \frac{d\mathbf{x}(\sigma)}{d\sigma} (w - \partial_z u)(\mathbf{x}(\sigma)) \right) w^{\frac{1}{2}} e^{\frac{u}{2}} \\ &\equiv \Psi(\mathbf{x}) w^{\frac{1}{2}} e^{\frac{u}{2}} \end{aligned} \quad (4.37)$$

where the path  $P$  is taken along  $\mathbb{R}^3$  with the vector  $\mathbf{k} = (0, 0, 1)$ . We will simply call  $\Psi(\mathbf{x})$  an open Wilson line because  $w - \partial_z u$  is a gauge field as can be seen from (3.4).

One can check that the result (4.37) is consistent with the previous ones. It is obvious that the frame rotation (4.34) reproduces (4.20) for the real heaven case with  $A_3(\mathbf{x}) = 0$  and the relation  $w = \partial_z u$  trivializes the open Wilson line in (4.37). For the Gibbons-Hawking metric with  $u = 0$  but  $A_3(\mathbf{x}) \neq 0$ , we don't have to take a frame rotation at the outset. Nevertheless we can solve (2.7) in a rotated frame like (4.34) too. In such a rotated frame, we get an extra factor  $w$  in (4.36) due to the frame rotation which must be canceled out by the open Wilson line in  $\lambda$ . In this respect, the LeBrun metric (3.1) is a kind of mixture of these two metrics.

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<sup>8</sup>Unfortunately we cannot make a similar reduction for the Gibbons-Hawking metric. The system (4.2) consists of vector fields on  $\mathbb{R}^3 \times \mathbb{S}^1$  and the Lie algebra of vector fields on  $\mathbb{S}^1$  is the Virasoro algebra [52]. But the vector field in the Virasoro algebra is not a Hamiltonian vector field because  $\mathbb{S}^1$  is not a symplectic manifold. So it is required that the Gibbons-Hawking metric resides in a four-dimensional symplectic manifold.

The vector fields  $V_a = \lambda E_a \in \Gamma(TM)$  for the LeBrun metric (3.1) are then given by (after dropping the prime)

$$\begin{aligned} V_1 &= \Psi(\mathbf{x}) \left( \frac{\partial}{\partial x} - A_1 \frac{\partial}{\partial \tau} \right), \\ V_2 &= \Psi(\mathbf{x}) \left( \frac{\partial}{\partial y} - A_2 \frac{\partial}{\partial \tau} \right), \\ V_3 &= e^{\frac{u}{2}} \Psi(\mathbf{x}) \left( \cos \frac{\phi}{2} \left( \frac{\partial}{\partial z} - A_3 \frac{\partial}{\partial \tau} \right) - w \sin \frac{\phi}{2} \frac{\partial}{\partial \tau} \right), \\ V_4 &= e^{\frac{u}{2}} \Psi(\mathbf{x}) \left( \sin \frac{\phi}{2} \left( \frac{\partial}{\partial z} - A_3 \frac{\partial}{\partial \tau} \right) + w \cos \frac{\phi}{2} \frac{\partial}{\partial \tau} \right). \end{aligned} \quad (4.38)$$

Note that  $\partial_x \Psi(\mathbf{x}) = \partial_y \Psi(\mathbf{x}) = 0$  and  $\partial_z \Psi(\mathbf{x}) = -\frac{1}{2}(w - \partial_z u) \Psi(\mathbf{x})$ . A straightforward calculation shows that the vector fields in (4.38) are not divergence-free unlike the previous self-dual metrics. Instead they obey the relation

$$\begin{aligned} \partial_\mu V_1^\mu &= \partial_\mu V_2^\mu = 0, \\ \partial_\mu V_3^\mu &= (w - \partial_z u) \Psi(\mathbf{x}) e^{\frac{u}{2}} \cos \frac{\phi}{2}, \\ \partial_\mu V_4^\mu &= (w - \partial_z u) \Psi(\mathbf{x}) e^{\frac{u}{2}} \sin \frac{\phi}{2}. \end{aligned} \quad (4.39)$$

One may notice that the divergence-free condition is violated even for the Gibbons-Hawking metric after the frame rotation (4.34) although the real heaven case was not affected by it. However, it should not be taken as a surprise because this divergence-free condition is not preserved under a general internal rotation of basis vectors.

It will be worthwhile to recall that the bottom-up approach implicitly assumes the on-shell condition. This means that we have to assume the Toda equation (3.2) and its linearization (3.3) for the solution (3.1) from the outset. As we remarked in the second paragraph of section 3, a linear deviation from a Toda point,  $u(\mathbf{x}) = u_t(\mathbf{x}) + w(\mathbf{x})$ , still satisfies the Toda equation (3.2) as long as  $u_t(\mathbf{x})$  is a solution of (3.2). Using this property, we may choose a particular path  $P$  in order to define the open Wilson line (4.37) such that, at the end point of the path where  $\mathbf{x}(\sigma = 1) = (x, y, z)$ ,

$$\partial_x \left( u(\mathbf{x}) - \int^z w(\mathbf{x}) dz \right) = \partial_y \left( u(\mathbf{x}) - \int^z w(\mathbf{x}) dz \right) = 0, \quad \partial_z u(\mathbf{x}) = w(\mathbf{x}). \quad (4.40)$$

Such a path  $P$  can be chosen with impunity because the path  $P$  obeying (4.40) is consistent with the Toda equation (3.2) and its linearization (3.3):

$$\partial_x^2 u + \partial_y^2 u + \partial_z^2 e^u = \int^z \left( \partial_x^2 w + \partial_y^2 w + \partial_z^2 (e^u w) \right) dz = 0. \quad (4.41)$$

For this reason, we call a path  $P$  obeying (4.40) on-shell path.<sup>9</sup> Adopting the same prescription of path ordering in noncommutative gauge theory [40, 41, 42], we will consider the Lie algebra of vector fields consisting of all local functions attached at one end of the open

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<sup>9</sup>We have chosen the end point  $\mathbf{x}(\sigma = 1) = (x, y, z)$  for the differentiation of  $\Psi(\mathbf{x})$ . We may equally choose the other end point  $\mathbf{x}(\sigma = 0)$  for a differentiation point, as well. Hence the on-shell condition (4.40) actually must be imposed on both ends. Then it means that the  $U(1)$  fiber represented by  $w - \partial_z u$  is pinched off at two end points of the open Wilson line in (4.37), which is very similar to the situation of Figure 1 in [34]. Therefore the open Wilson line represents a two-cycle in the LeBrun metric.

Wilson line with the on-shell condition (4.40) being satisfied. Then (4.39) suggests that, on the on-shell path, the vector fields  $V_a$  for the LeBrun metric are actually divergence-free.

The LeBrun metric (3.1) is a four-dimensional Kähler metric with a vanishing Ricci scalar. As we explained in the last part of section 2, the emergent gravity implies that such a scalar-flat Kähler manifold can emerge from pure Maxwell theory on a symplectic manifold. If the LeBrun metric is an example of such a case, it has to satisfy (2.30). Now we will show that the LeBrun metric (3.1) certainly obeys the scalar-flat condition (2.30).

For this purpose, let us determine the coefficients  $g_{ab}^c$  in the infinite-dimensional Lie algebra (2.11). In this calculation, we will use the result (3.4) for the U(1) field strength but we will not assume the on-shell condition (4.40) which will be imposed at the very last stage. A straightforward though tedious calculation shows that

$$[V_1, V_2] = \Psi(\mathbf{x}) e^{\frac{u}{2}} \partial_z \log(we^u) \left( \sin \frac{\phi}{2} V_3 - \cos \frac{\phi}{2} V_4 \right), \quad (4.42)$$

$$[V_3, V_4] = -\Psi(\mathbf{x}) e^{\frac{u}{2}} \partial_z \log(we^u) \left( \sin \frac{\phi}{2} V_3 - \cos \frac{\phi}{2} V_4 \right), \quad (4.43)$$

$$\begin{aligned} [V_1, V_3] = & \frac{1}{2} \Psi(\mathbf{x}) e^{\frac{u}{2}} (w - \partial_z u) \cos \frac{\phi}{2} V_1 + \frac{1}{2} \Psi(\mathbf{x}) \left( \partial_x u V_3 + \int^z \partial_y w dz V_4 \right) \\ & + \Psi(\mathbf{x}) \left( \partial_x \log w \sin \frac{\phi}{2} - \partial_y \log w \cos \frac{\phi}{2} \right) \left( \sin \frac{\phi}{2} V_3 - \cos \frac{\phi}{2} V_4 \right), \end{aligned} \quad (4.44)$$

$$\begin{aligned} [V_2, V_4] = & \frac{1}{2} \Psi(\mathbf{x}) e^{\frac{u}{2}} (w - \partial_z u) \sin \frac{\phi}{2} V_2 + \frac{1}{2} \Psi(\mathbf{x}) \left( \partial_y u V_4 + \int^z \partial_x w dz V_3 \right) \\ & + \Psi(\mathbf{x}) \left( \partial_x \log w \sin \frac{\phi}{2} - \partial_y \log w \cos \frac{\phi}{2} \right) \left( \sin \frac{\phi}{2} V_3 - \cos \frac{\phi}{2} V_4 \right), \end{aligned} \quad (4.45)$$

$$\begin{aligned} [V_1, V_4] = & \frac{1}{2} \Psi(\mathbf{x}) e^{\frac{u}{2}} (w - \partial_z u) \sin \frac{\phi}{2} V_1 + \frac{1}{2} \Psi(\mathbf{x}) \left( \partial_x u V_4 - \int^z \partial_y w dz V_3 \right) \\ & - \Psi(\mathbf{x}) \left( \partial_x \log w \cos \frac{\phi}{2} + \partial_y \log w \sin \frac{\phi}{2} \right) \left( \sin \frac{\phi}{2} V_3 - \cos \frac{\phi}{2} V_4 \right), \end{aligned} \quad (4.46)$$

$$\begin{aligned} [V_2, V_3] = & \frac{1}{2} \Psi(\mathbf{x}) e^{\frac{u}{2}} (w - \partial_z u) \cos \frac{\phi}{2} V_2 + \frac{1}{2} \Psi(\mathbf{x}) \left( \partial_y u V_3 - \int^z \partial_x w dz V_4 \right) \\ & + \Psi(\mathbf{x}) \left( \partial_x \log w \cos \frac{\phi}{2} + \partial_y \log w \sin \frac{\phi}{2} \right) \left( \sin \frac{\phi}{2} V_3 - \cos \frac{\phi}{2} V_4 \right). \end{aligned} \quad (4.47)$$

From the above results, one can easily read off the coefficients  $g_{ab}^c$  in the Lie algebra (2.11). Using the definition (2.28), one can deduce the following relations

$$\begin{aligned} \rho_1 + \Psi_1 &= \partial_x \left( u(\mathbf{x}) - \int^z w(\mathbf{x}) dz \right) \Psi(\mathbf{x}), \\ \rho_2 + \Psi_2 &= \partial_y \left( u(\mathbf{x}) - \int^z w(\mathbf{x}) dz \right) \Psi(\mathbf{x}), \\ \rho_3 + \Psi_3 &= -(w - \partial_z u) \Psi(\mathbf{x}) e^{\frac{u}{2}} \cos \frac{\phi}{2}, \\ \rho_4 + \Psi_4 &= -(w - \partial_z u) \Psi(\mathbf{x}) e^{\frac{u}{2}} \sin \frac{\phi}{2}. \end{aligned} \quad (4.48)$$

Interestingly, the divergence equation (4.39) indicates that  $\rho_3 + \Psi_3 = -\partial_\mu V_3^\mu$  and  $\rho_4 + \Psi_4 = -\partial_\mu V_4^\mu$ . In (4.48) and above equations, we are implicitly assuming the prescription of path ordering described below (4.41) to attach local functions at one end of  $\Psi(\mathbf{x})$ .

As we have justified before, we can choose a path  $P$  in order to satisfy the on-shell condition (4.40) to define the open Wilson line (4.37). Strictly speaking, it is actually

required because the two functions  $u(\mathbf{x})$  and  $w(\mathbf{x})$  must satisfy (3.2) and (3.3), respectively. So far we have not imposed the on-shell condition (4.40) anywhere. After applying the on-shell condition (4.40) to (4.48), we can immediately deduce that the scalar flat condition

$$\rho_a = -\Psi_a \quad (4.49)$$

is truly satisfied. This fact demonstrates that the LeBrun metric (3.1) can arise from pure Maxwell theory on a four-dimensional symplectic manifold whose equations of motion are given by (2.22).

We will not try to solve (4.38) to obtain symplectic gauge fields for the LeBrun metric. It may be premature before getting them for the Gibbons-Hawking metric (3.6) because the LeBrun metric (3.1) contains (3.6) and (3.8) as particular cases. But we will get back to the problem in the near future.

## 5. Conclusion

Let us recapitulate the lesson perceived from the bottom-up approach of emergent gravity. In the top-down approach of emergent gravity, we have symplectic gauge fields (or non-commutative gauge fields at very short distances) and their dynamical equations of motion. The most accessible frame in this case is the Darboux frame where symplectic gauge fields are defined by solving (1.3). But we are not compelled to reside in the Darboux frame as we already emphasized in the footnote 4. In principle we can formulate the gauge theory in an  $\Omega$ -frame (using the notation of the footnote 4). The gauge theory in this case will be described by the Poisson bracket (1.18) with a nontrivial Poisson tensor  $\Theta^{\mu\nu}(x)$  like (1.15) and the corresponding symplectic gauge fields are then defined by (1.22). It was previously argued (see the paragraph around (2.26) in [9]) that the resulting gauge theory is equivalent to the gauge theory on a curved space with a canonical Poisson tensor. Under both circumstances (either with a nontrivial Poisson tensor on a flat space or with a canonical Poisson tensor on a curved space), the construction of the full noncommutative gauge theory is a challenging problem. Thus the Darboux frame provides the most rudimentary gadget to formulate emergent gravity. But a caveat is that we cannot make a direct comparison with a gravitational metric since the gravitational metric is represented in a general coordinate system which is not necessarily in the Darboux frame, as we already emphasized in section 1. If we could formulate gauge theory and its emergent gravity in a general  $\Omega$ -frame, it would be possible to directly get gravitational metrics in Einstein gravity from the top-down approach.

In the bottom-up approach of emergent gravity explored in this paper, we start with a gravitational metric given on a Riemannian manifold  $M$ . We can solve (2.6) to determine the Weyl factor  $\lambda$  for a given metric and then identify the vector fields in (1.21) via the relationship (1.25). We found that the Weyl factor  $\lambda$  for a general metric contains a dipole-like object (which we dubbed an open Wilson line according to the similarity appearing in noncommutative gauge theory). Nevertheless, either gravitational metrics or tetrads are still described by local functions according to the relation (1.25). An intriguing point is that it seems unnecessary to introduce the dipole-like Weyl factor for gravitational



instantons and we believe that this may be applicable to all kinds of gravitational instantons. However it turned out that symplectic gauge fields in a general coordinate system are nonlocal functions even in commutative limit but with a Poisson structure defined thereto. The appearance of nonlocality may be expected due to the following reasons. In noncommutative gauge theories, local gauge invariant observables do not exist since we can effect a spatial translation by a gauge transformation [42]. The interrelation between a gauge transformation and a spatial translation still persists in the commutative limit as we remarked in the footnote 7. Therefore we cannot construct a local gauge invariant observable using symplectic gauge fields. This is also consistent with the idea of emergent gravity. In general relativity there exist no local gauge invariant observables either, as translations are equivalent to general coordinate transformations. Thereby, from this point of view, the emergence of dipole-like objects may be quite natural when we try to define symplectic gauge fields from a gravitational metric.

In spite of some difficulty to treat nonlocal objects such as (4.9) and (4.37), the bottom-up approach of emergent gravity nicely confirms the results of the top-down approach and elucidates many important aspects on emergent gravity as was summarized above. For example, the bottom-up approach renders a novel verification of the equivalence between gravitational instantons and symplectic  $U(1)$  instantons [45]. In particular, the real heaven case presents a paragon of the bottom-up approach by successfully producing the solution (4.33) of the BPS equation (4.31). If one tries to solve the equation (4.31) directly, it would be difficult to embody a solution. We think it already demonstrates a sound aspect of the bottom-up approach for emergent gravity. In addition to the explicit solution, a more noteworthy success is to verify that the LeBrun metric (3.1) is a solution of pure Maxwell theory on a four-dimensional symplectic manifold whose equations of motion are given by (2.22). It also constitutes a nontrivial check of the formula (2.24) derived in [9]. Therefore, if we can extract symplectic gauge fields from the vector fields in (4.38) by solving (1.24), it will constitute a very general class of solutions for noncommutative gauge theory and quantum gravity. We hope to open that direction with the work [44].

One may be tempted to apply the bottom-up approach of emergent gravity to (Euclidean) Schwarzschild black hole. The Euclidean Schwarzschild black hole solution describes a Ricci-flat manifold [55]. But it is not a Kähler manifold. So it does not admit a natural symplectic structure available in the  $\Omega$ -frame (1.17). The best alternative choice is to utilize the (anti-)self-dual harmonic two-forms on the space (see eq.(3) in [56]) and define a Poisson algebra determined by the self-dual harmonic two-form. However a magnetic mass (and an electric mass) at the origin seems to bring about the violation of Jacobi identity of the underlying Poisson algebra similar to the situation of a charged quantum particle in the presence of a magnetic monopole [57]. Therefore the Schwarzschild black hole remains a challenging goal for the top-down as well as the bottom-up approaches of emergent gravity.

So far we have implicitly assumed that fluctuations  $F$  in (1.1) have no homogeneous sink on vacuum. In other words, we have exclusively considered local fluctuations so that  $|F| \rightarrow 0$  at asymptotic infinity. In this case the Darboux frame is defined by a coordinate transformation  $\phi \in \text{Diff}(M)$  obeying  $\phi^*(B + F) = B$  as in (1.3). But we may consider a

general kind of fluctuations allowing a homogeneous condensate on vacuum. This means that fluctuations  $F$  will change even the asymptotic vacuum structure and so the Darboux transformation  $\phi \in \text{Diff}(M)$  instead is defined by  $\phi^*(B+F) = B + \langle F(|x| \rightarrow \infty) \rangle_{\text{vac}} \equiv B'$ . If  $\text{rank}(B) = \text{rank}(B')$ , we may introduce a nowhere vanishing function  $f$  such that  $B' = fB$ . Therefore, to describe such situation, we need to introduce an almost symplectic manifold  $(M, \Omega)$  where the two-form  $\Omega$  is nondegenerate but not necessarily closed and, in particular, is locally conformal to a symplectic form  $\omega$ . Such an almost symplectic manifold is known as a locally conformal symplectic manifold [58, 59]. On a locally conformal symplectic manifold  $(M, \Omega)$ , there exists an open covering  $U_\alpha$  of  $M = \bigcup_\alpha U_\alpha$  and a smooth positive function  $f_\alpha$  on each  $U_\alpha$  such that  $f_\alpha \Omega|_{U_\alpha} \equiv \Omega_\alpha$  is symplectic on  $U_\alpha$ . This is equivalent to the existence of a closed one-form  $\eta$ , the so-called Lee form [58], such that

$$d\Omega = \eta \wedge \Omega. \quad (5.1)$$

When  $\eta$  vanishes identically, we recover the symplectic two-form  $\Omega$ . And any Hamiltonian vector field  $X$  on a locally conformal symplectic manifold satisfies [59]

$$\mathcal{L}_X \Omega_\alpha = k \Omega_\alpha \quad (5.2)$$

with a constant  $k$ . It turns out [60] that it is necessary to introduce such a locally conformal symplectic structure to describe the epoch of cosmic inflation of our universe. If so, the locally conformal symplectic structure might play an important role for the birth and the evolution of our Universe.

In conclusion, we are yet to invite the most important two players—the Schwarzschild black hole and the cosmic inflation—to the league of emergent gravity. Confrontations with them will certainly help us lift the veil of quantum gravity.

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